

# Mean-field model for Josephson oscillation in a Bose-Einstein condensate on an one-dimensional optical trap

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**Abstract.** Using the axially-symmetric time-dependent Gross-Pitaevskii equation we study the phase coherence in a repulsive Bose-Einstein condensate (BEC) trapped by a harmonic and an one-dimensional optical lattice potential to describe the experiment by Cataliotti *et al.* on atomic Josephson oscillation [Science **293**, 843 (2001)]. The phase coherence is maintained after the BEC is set into oscillation by a small displacement of the magnetic trap along the optical lattice. The phase coherence in the presence of oscillating neutral current across an array of Josephson junctions manifests in an interference pattern formed upon free expansion of the BEC. The numerical response of the system to a large displacement of the magnetic trap is a classical transition from a coherent superfluid to an insulator regime and a subsequent destruction of the interference pattern in agreement with the more recent experiment by Cataliotti *et al.* [New J. Phys. **5**, 71 (2003)].

**PACS.** 03.75.-b Matter waves – 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations – 03.75.Kk Dynamic properties of condensates; collective and hydrodynamic excitations, superfluid flow

## 1 Introduction

The observation of an oscillating Josephson current across the boundaries of a one-dimensional array of potential wells, usually generated by a standing-wave laser field and commonly known as an optical lattice potential, in a trapped cigar-shaped Bose-Einstein condensate (BEC) by Cataliotti *et al.* [1] is a clear manifestation of macroscopic quantum phase coherence. So far the Josephson effect has been confirmed in superconductors with charged electrons and in liquid helium [2].

The recent experimental observation of BEC in trapped alkali-metal atoms [3] has offered new possibility of the confirmation of Josephson effect in neutral quantum fluids with an array of quasi one-dimensional Josephson junctions not realizable in superconductors. The experimental loading of a cigar-shaped BEC in both one- [4–6] and three-dimensional [7] optical lattice potentials has allowed the study of quantum phase effects on a macroscopic scale such as interference of matter waves [8]. There have been several theoretical studies on different aspects of a BEC in a one- [9] as well as three-dimensional [10] optical lattice potentials. The phase coherence between different sites of a trapped BEC on an optical lattice has been established in recent experiments [1, 4, 5, 7] through the formation of distinct interference pattern when the traps are removed.

Cataliotti *et al.* [1, 11] have provided a quantitative measurement of the formation and evolution of interference pattern upon free expansion of a cigar-shaped trapped BEC of repulsive Rb atoms on an optical lattice and harmonic potentials after the removal of the combined traps. The phase coherence in a BEC trapped in a standing-wave optical-lattice is responsible for the formation of a distinct interference pattern upon free expansion as observed in several recent experiments [1, 4, 5, 7, 11]. Cataliotti *et al.* [1] also continued their investigation to a BEC oscillating on the optical lattice *via* quantum tunneling and found that the phase coherence between different sites is maintained during oscillation initiated by a sudden shift of the magnetic trap along the optical axis.

The phase-coherent BEC on the optical lattice is a quantum superfluid [7] and the atoms in it move freely from one optical site to another by quantum tunneling. However, the classical movement is prohibited through the high optical potential traps. It has been demonstrated in a recent experiment by Greiner *et al.* [7] that, as the optical potential traps are made much too higher, the quantum tunneling of atoms from one optical site to another is stopped resulting in a loss of superfluidity and phase coherence in the BEC. Consequently, no interference pattern is formed upon free expansion of such a BEC which is termed a Mott insulator state. This phenomenon represents a quantum phase transition (with energy nonconservation in the tunneling process) and cannot be accounted

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for in a classical mean-field model based on the Gross-Pitaevskii (GP) equation [12].

Following a suggestion by Smerzi *et al.* [13], more recently Cataliotti *et al.* [14] have demonstrated in a novel experiment the loss of phase coherence and superfluidity in a BEC trapped in an optical-lattice and harmonic potentials when the center of the harmonic potential is suddenly displaced along the optical lattice through a distance larger than a critical value. Then a modulational instability takes place in the BEC and it cannot reorganize itself quickly enough and the phase coherence and superfluidity of the BEC are destroyed. The resulting motion of the condensate is not oscillatory in nature. The loss of phase coherence is manifested in the destruction of the interference pattern upon free expansion. However, for displacements smaller than the critical distance the BEC can reorganize itself and the phase coherence and superfluidity are maintained [1, 14]. Recently, a new classical mechanism for the loss of superfluidity of a BEC arising from a nonlinear modulation of the scattering length has been suggested [15]. Distinct from the quantum phase transition observed by Greiner *et al.* [7], these modulational instabilities responsible for the destruction of phase coherence are classical in nature and can be described [13, 14] by the mean-field model. Hence in the present paper we present a mean-field description of the experiments by Cataliotti *et al.* [1, 14] to see how well and how far it can describe the observed phenomena. Specifically, we consider the numerical solution of the axially-symmetric GP equation [12] in an optical and a harmonic trap.

Cataliotti *et al.* [1, 14] provided a theoretical account of their study using the tight-binding approximation for the full wave function in the presence of the periodic optical potential wells. Also, there has been a preliminary attempt to explain some features of this experiment using one-dimensional mean-field models [11, 16]. In reference [16] a dynamical solution of one-dimensional GP equation was used; whereas in reference [11] an one-dimensional model of interference was developed using superposition of analytical matter waves, which is reasonable in the absence of nonlinear atomic interaction. Although, the tight-binding approximation and these one-dimensional models could be reasonable for the study of some aspects of the experiment of Cataliotti *et al.* [1, 14], here we compare the results with the complete solution of the three-dimensional mean-field Hamiltonian *via* the nonlinear GP equation [12].

In Section 2 we present the mean-field model based on the axially-symmetric time-dependent nonlinear GP equation. In Section 3 we present the numerical results and finally, in Section 4 we present the conclusions.

## 2 Mean-field model

The time-dependent BEC wave function  $\Psi(\mathbf{r}; t)$  at position  $\mathbf{r}$  and time  $t$  is described by the following mean-field nonlinear GP equation [12]

$$\left[ -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) + gN |\Psi(\mathbf{r}; t)|^2 \right] \Psi(\mathbf{r}; t) = 0, \quad (1)$$

where  $m$  is the mass and  $N$  the number of atoms in the condensate,  $g = 4\pi\hbar^2 a/m$  the strength of interatomic interaction, with  $a$  the atomic scattering length. In the presence of the combined axially-symmetric and optical lattice traps  $V(\mathbf{r}) = \frac{1}{2}m\omega^2(\rho^2 + \nu^2 y^2) + V_{\text{opt}}$  where  $\omega$  is the angular frequency of the harmonic trap in the radial direction  $\rho$ ,  $\nu\omega$  that in the axial direction  $y$ , with  $\nu$  the aspect ratio, and  $V_{\text{opt}}$  is the optical lattice potential introduced later. The normalization condition is  $\int d\mathbf{r} |\Psi(\mathbf{r}; t)|^2 = 1$ .

In the axially-symmetric configuration, the wave function can be written as  $\Psi(\mathbf{r}, t) = \psi(\rho, y, t)$ , where  $0 \leq \rho < \infty$  is the radial variable and  $-\infty < y < \infty$  is the axial variable. Now transforming to dimensionless variables  $\hat{\rho} = \sqrt{2}\rho/l$ ,  $\hat{y} = \sqrt{2}y/l$ ,  $\tau = t\omega$ ,  $l \equiv \sqrt{\hbar/(m\omega)}$ , and  $\varphi(\hat{\rho}, \hat{y}; \tau) \equiv \hat{\rho} \sqrt{l^3/\sqrt{8}} \psi(\rho, y; t)$ , equation (1) becomes [17]

$$\left[ -i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \hat{\rho}^2} + \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} - \frac{\partial^2}{\partial \hat{y}^2} + \frac{1}{4} (\hat{\rho}^2 + \nu^2 \hat{y}^2) + \frac{V_{\text{opt}}}{\hbar\omega} - \frac{1}{\hat{\rho}^2} + 8\sqrt{2}\pi n \left| \frac{\varphi(\hat{\rho}, \hat{y}; \tau)}{\hat{\rho}} \right|^2 \right] \varphi(\hat{\rho}, \hat{y}; \tau) = 0, \quad (2)$$

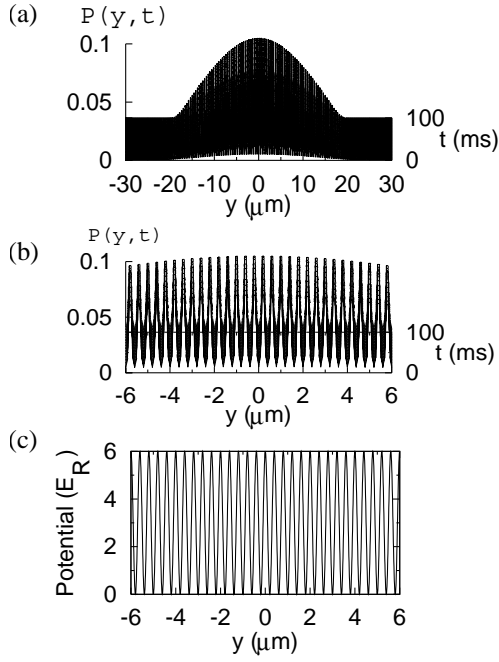
where  $n = Na/l$ . In terms of the one-dimensional probability  $P(y, t) \equiv 2\pi \int_0^\infty d\hat{\rho} |\varphi(\hat{\rho}, \hat{y}, \tau)|^2 / \hat{\rho}$ , the normalization of the wave function is given by  $\int_{-\infty}^\infty d\hat{y} P(y, t) = 1$ . The probability  $P(y, t)$  is useful in the study of the present problem under the action of the optical lattice potential, specially in the investigation of the formation and evolution of the interference pattern after the removal of the trapping potentials.

In the experiment of Cataliotti *et al.* [1] with repulsive  $^{87}\text{Rb}$  atoms in the hyperfine state  $F = 1$ ,  $m_F = -1$ , the axial and radial trap frequencies were  $\nu\omega = 2\pi \times 9$  Hz and  $\omega = 2\pi \times 92$  Hz, respectively, with  $\nu = 9/92$ . The optical potential created with the standing-wave laser field of wavelength  $\lambda = 795$  nm is given by  $V_{\text{opt}} = V_0 E_R \cos^2(k_L z)$ , with  $E_R = \hbar^2 k_L^2 / (2m)$ ,  $k_L = 2\pi/\lambda$ , and  $V_0 (< 12)$  the strength. For the mass  $m = 1.441 \times 10^{-25}$  kg of  $^{87}\text{Rb}$  the harmonic oscillator length  $l = \sqrt{\hbar/(m\omega)} = 1.126$   $\mu\text{m}$  and the present dimensionless length unit corresponds to  $l/\sqrt{2} = 0.796$   $\mu\text{m}$ . The present dimensionless time unit corresponds to  $\omega^{-1} = 1/(2\pi \times 92)$  s = 1.73 ms. Although we perform the calculation in dimensionless units using equation (2), we present the results in actual physical units using these conversion factors consistent with the experiment by Cataliotti *et al.* [1]. In terms of the dimensionless laser wave length  $\lambda_0 = \sqrt{2}\lambda/l \simeq 1$ , the dimensionless standing-wave energy parameter  $E_R/(\hbar\omega) = 4\pi^2/\lambda_0^2$ . Hence in dimensionless unit  $V_{\text{opt}}$  of equation (2) is

$$\frac{V_{\text{opt}}}{\hbar\omega} = V_0 \frac{4\pi^2}{\lambda_0^2} \left[ \cos^2 \left( \frac{2\pi}{\lambda_0} \hat{y} \right) \right]. \quad (3)$$

We solve equation (2) numerically using a split-step time-iteration method with the Crank-Nicholson discretization scheme described recently [18]. The time iteration is started with the known harmonic oscillator solution of equation (2) with  $n = 0$ :

$$\varphi(\hat{\rho}, \hat{y}) = [\nu/(8\pi^3)]^{1/4} \hat{\rho} e^{-(\hat{\rho}^2 + \nu^2 \hat{y}^2)/4}$$

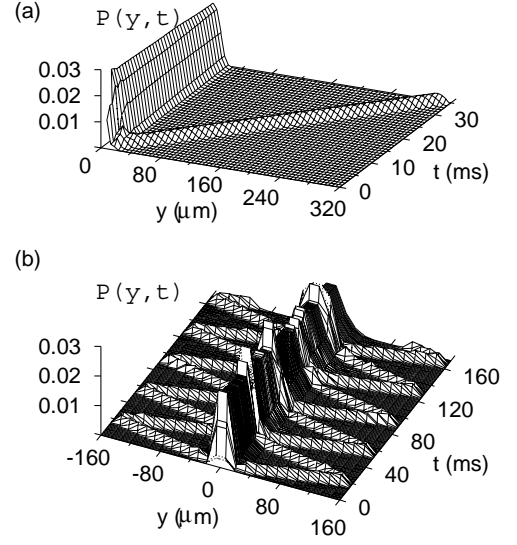


**Fig. 1.**  $P(y, t)$  vs.  $y$  and  $t$  for the ground-state BEC with  $n = 10$  and  $V_0 = 6$  for  $0 < t < 100$  ms and (a)  $-30 \mu\text{m} < y < 30 \mu\text{m}$  and (b)  $-6 \mu\text{m} < y < 6 \mu\text{m}$ . The optical potential for  $-6 \mu\text{m} < y < 6 \mu\text{m}$  is shown in (c). The harmonic oscillator potential is negligible on this scale.

with chemical potential  $\bar{\mu} = (1 + \nu/2)$  [17]. For a typical cigar-shaped condensate with  $\nu \simeq 0.1$  [1],  $\bar{\mu} \simeq 1$  is much smaller than the typical depth of the optical potential wells  $E_R/(\hbar\omega) = 4\pi^2/\lambda_0^2 \simeq 40$  so that  $\bar{\mu} \ll E_R/(\hbar\omega)$  and the passage of condensate atoms from one well to other can only proceed through quantum tunneling. The nonlinearity as well as the optical lattice potential parameter  $V_0$  are slowly increased by equal amounts in 10 000  $n$  steps of time iteration until the desired value of nonlinearity and optical lattice potentials are attained. Then, without changing any parameter, the solution so obtained is iterated 50 000 times so that a stable solution is obtained independent of the initial input and time and space steps. The solution then corresponds to the bound BEC under the joint action of the harmonic and optical traps.

### 3 Numerical results

First we consider the BEC formed on the optical lattice for a specific nonlinearity. In the present study we take nonlinearity  $n = 10$  and optical lattice strength  $V_0 = 6$  except in Figure 5 where we use a variable  $V_0$ . We consider the ground-state wave function in the combined harmonic and optical lattice potentials. The one-dimensional pattern in the axial  $y$ -direction is most easily illustrated from a consideration of the probability  $P(y, t)$  at different times. In Figure 1a we plot the frontal view of  $P(y, t)$  for  $0 < t < 100$  ms and  $-30 \mu\text{m} < y < 30 \mu\text{m}$ . In this interval of  $y$ , there are 150 wells of the optical potential and



**Fig. 2.**  $P(y, t)$  vs.  $y$  and  $t$  for the BEC of figure 1 after the removal of combined traps at  $t = 0$  for a lattice defined by  $\rho \leq 25 \mu\text{m}$  and (a)  $-320 \mu\text{m} < y < 320 \mu\text{m}$  and (b)  $-160 \mu\text{m} < y < 160 \mu\text{m}$ .

as many maxima and minima in  $P(y, t)$ , which cannot be visualized clearly in Figure 1a. In the actual experiment 200 wells were typically populated, which corresponds to a larger condensate than considered in this numerical simulation. For the limitation in computer processing time we had to stick to a smaller condensate. In Figure 1b we show a close-up of Figure 1a for  $-6 \mu\text{m} < y < 6 \mu\text{m}$  containing 30 wells. The corresponding optical potential is shown in Figure 1c, which clearly shows the 30 wells. From Figure 1b one can count 30 maxima and 30 minima in probability  $P(y, t)$ .

As the present calculation is performed with the full wave function without approximation, phase coherence among different wells of the optical lattice is automatically guaranteed. As a result when the condensate is released from the combined trap, a matter-wave interference pattern is formed in few milliseconds. The atom cloud released from one lattice site expand, and overlap and interfere with atom clouds from neighboring sites to form the robust interference pattern due to phase coherence. No interference pattern can be formed without phase coherence. The pattern consists of a central peak at  $y = 0$  and two symmetrically spaced peaks, each containing about 10% of total number of atoms, moving apart from the central peak [1, 11].

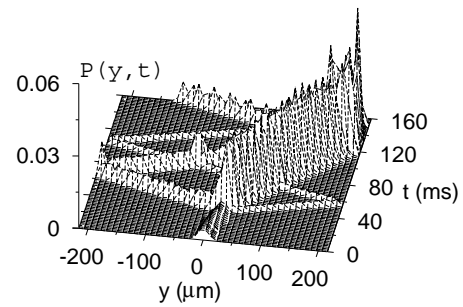
The simulation of the formation of the interference pattern is performed by loading the preformed condensate of Figure 1 on two lattices with  $\rho \leq 25 \mu\text{m}$  and (a)  $320 \mu\text{m} \geq y \geq -320 \mu\text{m}$ , and (b)  $160 \mu\text{m} \geq y \geq -160 \mu\text{m}$  which will permit the study of the evolution of the interference pattern on a large interval of space and time. The interference pattern is formed by suddenly removing the combined traps at time  $t = 0$ . The time evolution of the system is best illustrated *via*  $P(y, t)$  and we plot in Figures 2a and 2b  $P(y, t)$  vs.  $y$  and  $t$  for lattices (a) and (b), respectively.

The dynamics is symmetric about  $y = 0$  and only  $P(y, t)$  for positive  $y$  is plotted in Figure 2a. In these plots we can clearly see the central condensate and the moving interference peak(s). The peaks spread unobservably slowly as they propagate, even after reflection from the boundary or after crossing each other. The phase coherence between the components of BEC at different sites of optical lattice is responsible for the generation of the interference pattern with very little or practically no spreading. Without the initial phase coherence over a large number of lattice sites, a repulsive condensate in the absence of a trap will disappear in few milliseconds [19]. Each of the moving interference peaks is similar to atom laser [4, 20] which can be used in the scattering of two coherent BECs and other purposes.

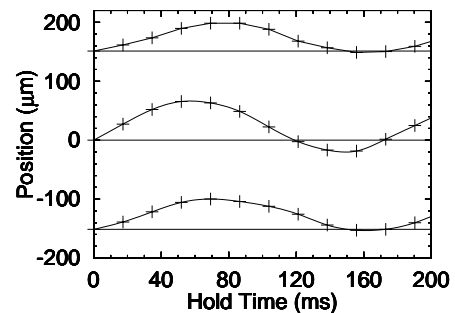
We have also examined the wave function  $\varphi(\hat{\rho}, \hat{y}, t)$  at different times (not reported here). We find that there is virtually no spreading of the wave function in the axial  $y$  direction during few hundred milliseconds. The phase coherence in the axial direction due to the optical lattice is responsible for the localization of the peaks.

Next we consider an oscillating BEC in the combined harmonic and optical traps. If we suddenly displace the magnetic trap along the lattice axis by a small distance after the formation of the BEC in the combined trap, the condensate will be out of equilibrium and start to oscillate. As the height of the potential-well barriers on the optical lattice is much larger than the energy of the system, the atoms in the condensate will move by tunneling through the potential barriers. This fluctuating transfer of Rb atoms across the potential barriers is due to Josephson effect in a neutral quantum liquid. The experiment of Cataliotti *et al.* [1, 11] demonstrates that the phase coherence between different wells of the condensate is maintained during this mass transfer and a matter-wave interference pattern with three peaks is formed after the removal of the joint trap. The peaks of the expanded condensate oscillate in phase, thus showing that the quantum mechanical phase coherence is maintained over the entire condensate. They studied this problem experimentally in some detail by varying the time of oscillation of the BEC (hold time) before removing the combined trap [1].

To see if the present simulation can represent the essential features of the phase coherence of the oscillating BEC, we load the GP equation with the BEC of the combined harmonic oscillator and optical traps on a lattice defined by  $200 \mu\text{m} \geq y \geq -200 \mu\text{m}$  and  $\rho \leq 25 \mu\text{m}$  and suddenly displace the harmonic trap along the optical axis by  $25 \mu\text{m}$ . The BEC starts to oscillate and we allow the oscillation to evolve through a certain interval of time, called hold time, before the removal of the combined traps. The interference pattern is observed after some time of free expansion and the positions of the interference peaks are noted. In Figure 3 we plot the one-dimensional probability  $P(y, t)$  vs.  $y$  and  $t$  after an initial evolution of the oscillation during 35 ms and observe the interference pattern for 160 ms. In this case, unlike in Figures 2, the large central peak does not stay at rest and the sizes and positions of the two smaller peaks are not symmetrical around  $y = 0$ .



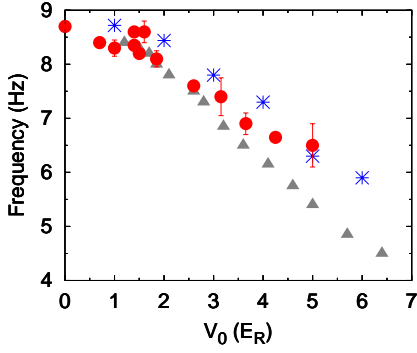
**Fig. 3.**  $P(y, t)$  vs.  $y$  and  $t$  for an oscillating BEC on optical lattice after a displacement of the magnetic trap through  $25 \mu\text{m}$  along the optical axis and upon the removal of the combined traps at  $t = 35$  ms (hold time).



**Fig. 4.** Center of mass positions of the three interference peaks of the expanded condensate after 20 ms of free expansion vs. hold time of the oscillating BEC. The magnetic trap is displaced through a distance of  $25 \mu\text{m}$ . The + symbols denote the results of simulation which are joined by full lines to show the correlated oscillation of the three peaks.

The clear formation of the interference pattern with very little spreading even after reflection at the boundaries and its propagation for more than 160 ms is noted in Figure 3 which confirms the phase coherence.

To study the phase coherence in detail we plot in Figure 4 the positions of the expanded interference peaks after 20 ms of free expansion for different hold times of oscillation in the displaced harmonic potential. We find that the interference peaks oscillate in phase showing the phase coherence in the oscillating BEC. Similar oscillation was also observed in the experiment of Cataliotti *et al.* [1]. From Figure 4 we find that the period of this oscillation is about 170 ms corresponding to a frequency of 5.9 Hz, which is very close to the experimental result exhibited in Figure 3 of reference [1]. To make a more complete comparison with Figure 3 of reference [1] we calculated the frequency of atomic current in the array of Josephson junctions for different  $V_0$  and the results are shown in Figure 5 where we plot the present frequencies as well as those of the experiment of Cataliotti *et al.* [1] and of their tight-binding calculation. From Figure 5 we see that the complete solution of the GP equation has led to results in agreement with the experiment of Cataliotti *et al.* The agreement of the present calculation in Figure 5 performed with a smaller condensate with experiment demonstrates



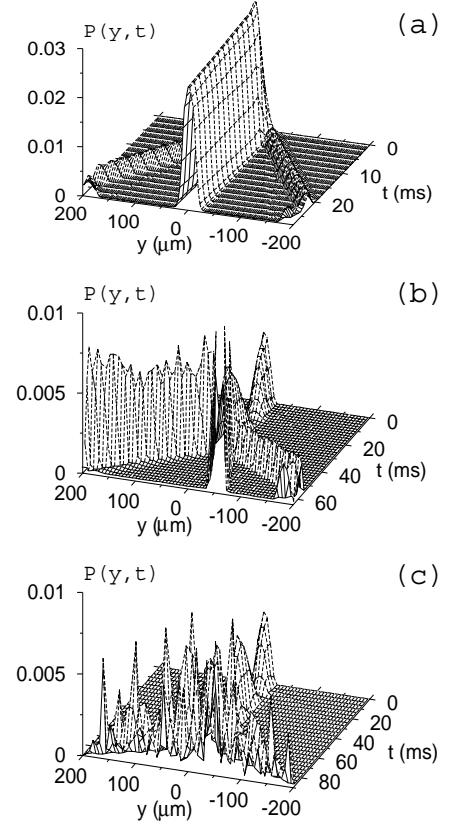
**Fig. 5.** The frequency of the atomic current in the array of Josephson junctions as a function of optical lattice strength: (●) with error bar – experiment of Cataliotti *et al.* [1]; (▲) tight binding calculation [1]; (★) present calculation.

that the frequency of atomic current is mostly determined by the strength of the optical lattice strength and is reasonably independent of the size of the condensate.

Finally, we consider the destruction of superfluidity in the condensate when the center of the magnetic trap is displaced along the optical lattice by a distance larger than the critical distance and the BEC is allowed to stay in this displaced trap for an interval of time (hold time). In this case the BEC does not execute an oscillatory motion but its center moves very slowly towards the new center of the magnetic trap. The destruction of superfluidity and phase coherence for a larger hold time in the displaced trap manifests in the disappearance of the interference pattern upon free expansion as noted in experiment [14]. As in that experiment, we consider a displacement of the magnetic trap through  $120 \mu\text{m}$  and allow the condensate to freely expand for 27.8 ms after different hold times in the optical and displaced magnetic traps.

For numerical simulation we load the BEC of Figure 1 on a lattice with  $\rho \leq 25 \mu\text{m}$  and  $200 \mu\text{m} \geq y \geq -200 \mu\text{m}$  and study the its evolution after an initial displacement of  $120 \mu\text{m}$  of the magnetic trap for hold times 0, 35 ms, and 70 ms. As in the experiment no oscillatory motion of the BEC is noted in the displaced trap. The corresponding probability densities are plotted in Figures 6a, 6b, and 6c, respectively. For hold times 0 and 35 ms prominent interference pattern is formed upon free expansion as we can see in Figures 6a and 6b. In these cases three separate pieces of interference patterns corresponding to three distinct trails can be identified. However, as the hold time in the displaced trap increases the maxima of the interference pattern mixes up and finally for a hold time of 70 ms the interference pattern is completely destroyed as we find in Figure 6c in agreement with the experiment [14].

As the BEC is allowed to evolve for a substantial interval of time after a large displacement of the magnetic trap along the optical axis a dynamical instability of classical nature sets in and the system can not evolve maintaining the phase coherence [13,14]. This has been explicitly demonstrated in the present simulation which results in the destruction of the interference pattern.



**Fig. 6.**  $P(y, t)$  vs.  $y$  and  $t$  for a BEC on optical lattice after a sudden displacement of the magnetic trap through  $120 \mu\text{m}$  along the optical axis and upon the removal of the combined traps after hold times (a) 0, (b) 35 ms, and (c) 70 ms. The time evolution is stopped upon 27.8 ms of free expansion after the removal of the combined traps.

## 4 Conclusion

In conclusion, to understand theoretically the experiments by Cataliotti *et al.* [1,14], we have studied in detail the phase coherence along a cigar-shaped condensate loaded in a combined axially-symmetric harmonic trap and optical lattice trap using the solution of the mean-field GP equation. Upon removal of the combined traps, the formation of an interference pattern clearly demonstrates the phase coherence over a very large number of optical lattice sites. Each of the moving interference peaks formed of coherent matter wave is similar to a atom laser observed experimentally [4,20]. The phase coherence along the optical lattice axis of the condensate is maintained even if the initial BEC is set into oscillation by suddenly shifting the harmonic trap along the optical axis through a small distance and keeping the BEC in the displaced trap for a certain hold time. This is clearly demonstrated by the noted correlated oscillation of the condensate peaks after free expansion for different hold times. The present mean-field model provides a proper account of the frequency of atomic current in the array of Josephson junctions in agreement with experiment [1].

However, if the initial displacement of the harmonic trap along the optical axis is larger than a critical value and the BEC is maintained in the displaced trap for a certain time, the phase coherence is destroyed. Consequently, after release from the combined trap no interference pattern is formed in agreement with experiment [14].

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## References

1. F.S. Cataliotti, S. Burger, C. Fort, P. Maddaloni, F. Minardi, A. Trombettoni, A. Smerzi, M. Inguscio, *Science* **293**, 843 (2001)
2. S.V. Pereverzev, S. Backhaus, A. Loshak, J.C. Davis, R.E. Packard, *Nature* **388**, 449 (1997)
3. M.H. Anderson, J.R. Ensher Jr, M.R. Matthews, C.E. Wieman, E.A. Cornell, *Science* **269**, 198 (1995); J.R. Ensher, D.S. Jin, M.R. Matthews, C.E. Wieman, E.A. Cornell, *Phys. Rev. Lett.* **77**, 4984 (1996); K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, W. Ketterle, *ibid.* **75**, 3969 (1995); D.G. Fried, T.C. Killian, L. Willmann, D. Landhuis, S.C. Moss, D. Kleppner, T.J. Greytak, *ibid.* **81**, 3811 (1998); C.C. Bradley, C.A. Sackett, J.J. Tollett, R.G. Hulet, *ibid.* **75**, 1687 (1995)
4. B.P. Anderson, M.A. Kasevich, *Science* **282**, 1686 (1998)
5. C. Orzel, A.K. Tuchman, M.L. Fenselau, M. Yasuda, M.A. Kasevich, *Science* **291**, 2386 (2001)
6. O. Morsch, M. Cristiani, J.H. Müller, D. Ciampini, E. Arimondo, *Phys. Rev. A* **66**, 021601 (2002); J.H. Müller, O. Morsch, M. Cristiani, D. Ciampini, E. Arimondo, e-print cond-mat/0211079
7. M. Greiner, O. Mandel, T. Esslinger, T.W. Hänsch, I. Bloch, *Nature* **415**, 39 (2002); M. Greiner, O. Mandel, T.W. Hänsch, I. Bloch, *ibid.* **419**, 51 (2002)
8. M.R. Andrews, C.G. Townsend, H.J. Miesner, D.S. Durfee, D.M. Kurn, W. Ketterle, *Science* **275**, 637 (1997)
9. Y.B. Band, M. Trippenbach, *Phys. Rev. A* **65**, 053602 (2002); W.M. Liu, W.B. Fan, W.M. Zheng, J.Q. Liang, S. T. Chui, *Phys. Rev. Lett.* **88**, 170408 (2002); M. Kramer, L. Pitaevskii, S. Stringari, *ibid.* **88**, 180404 (2002), K. Goral, L. Santos, M. Lewenstein, *ibid.* **88**, 170406 (2002); A.A. Sukhorukov, Y.S. Kivshar, *Phys. Rev. E* **65**, 036609 (2002); P. Massignan, M. Modugno, e-print cond-mat/0205516
10. S.K. Adhikari, P. Muruganandam, *Phys. Lett. A* **310**, 229 (2003)
11. P. Pedri, L. Pitaevskii, S. Stringari, C. Fort, S. Burger, F.S. Cataliotti, P. Maddaloni, F. Minardi, M. Inguscio, *Phys. Rev. Lett.* **87**, 220401 (2001)
12. F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999)
13. A. Smerzi, A. Trombettoni, P.G. Kevrekidis, A.R. Bishop, *Phys. Rev. Lett.* **89**, 170402 (2002)
14. F.S. Cataliotti, L. Fallani, F. Ferlaino, C. Fort, P. Maddaloni, M. Inguscio, *New J. Phys.* **5**, 71 (2003)
15. S.K. Adhikari, *Phys. Lett. A* **308**, 302 (2003)
16. S. Burger, F.S. Cataliotti, C. Fort, F. Minardi, M. Inguscio, M.L. Chiofalo, M.P. Tosi, *Phys. Rev. Lett.* **86**, 4447 (2001)
17. S. K. Adhikari, *Phys. Rev. E* **65**, 016703 (2002)
18. S.K. Adhikari, P. Muruganandam, *J. Phys. B* **35**, 2831 (2002); **36**, 409 (2003)
19. S.K. Adhikari, *Phys. Rev. A* **65**, 033616 (2002)
20. M.O. Mewes, M.R. Andrews, D.M. Kurn, D.S. Durfee, C.G. Townsend, W. Ketterle, *Phys. Rev. Lett.* **78**, 582 (1997); I. Bloch, T. Hänsch, T. Esslinger, *Phys. Rev. Lett.* **82**, 1686 (1999)